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AN AXIALLY SYMMETRIC CONTACT PROBLEM FOR A TRUNCATED SPHERE IN THE THEORY OF ELASTICITY†

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A method for reducing a problem in the theory of elasticity to a Hilbert boundary-value problem, which has been generalized by Vekua [1], is extended to a mixed axially symmetric problem for a truncated sphere, with a rigidly embedded spherical surface. The normal stresses at the cut are specified. The system of functional equations of this problem is transformed to a system of two singular equations which requires regularization. A contact problem concerning the impression of a punch which is circular in plan into the cut of the truncated sphere is then considered. The integral equation for this problem is reduced, using the method of paired equations, to a Fredholm integral equation of the second kind. © 1997 Elsevier Science Ltd. All rights reserved.

The contact problem of the torsion of an elastic truncated sphere with a clamped spherical surface by a rigid punch which is circular in plan view and located in the cut of the sphere has been studied previously [2–4]. An integral Mehler–Fock transformation on the real axis was used to derive the integral equation of this problem. Below, an integral Mehler–Fock transformation in the complex plane is used as in the case of the second fundamental boundary-value problem in the axially symmetric theory of elasticity for a spherical lens [1]. A truncated sphere is a special case of this.

1. We will consider the axially symmetric problem for an elastic truncated sphere with an embedded spherical surface. This truncated sphere is acted upon by a normal point force on its cut. We use the toroidal coordinates ξ, η (Fig. 1) which are associated with the cylindrical coordinates r, z, which have been divided by the radius of the section R_*, by the relations

$$r=\frac{\operatorname{sh}\xi}{\operatorname{ch}\xi+\cos\eta},\quad z=\frac{\sin\eta}{\operatorname{ch}\xi+\cos\eta} \tag{1.1}$$

We write the boundary conditions for the problem in the form (G is the shear modulus)

$$\begin{aligned}&\sigma_z/(2G)=-\delta(\xi-x),\quad \tau_{rz}=0\quad (\eta=0)\\ &u_r=u_z=0\quad (\eta=\alpha)\end{aligned} \tag{1.2}$$

We express the strains and stresses from (1.2) in terms of two harmonic Boussinesq functions Φ_n ($n = 1, 2$) using the formulae in [5] (ν is Poisson's ratio)

$$\begin{aligned}&\frac{u_r}{R_*}=\frac{\partial\Phi_1}{\partial r}-z\frac{\partial\Phi_2}{\partial r},\quad \frac{u_z}{R_*}=\frac{\partial\Phi_1}{\partial z}-z\frac{\partial\Phi_2}{\partial z}+\varepsilon_3\Phi_2\quad (\varepsilon_3=3-4\nu)\\ &\frac{\tau_{rz}}{2G}=\frac{\partial^2\Phi_1}{\partial r\partial z}-z\frac{\partial^2\Phi_2}{\partial r\partial z}+\varepsilon_1\frac{\partial\Phi_2}{\partial r}\quad (\varepsilon_1=1-2\nu)\\ &\frac{\sigma_z}{2G}=\frac{\partial^2\Phi_1}{\partial z^2}-z\frac{\partial^2\Phi_2}{\partial z^2}+\varepsilon_2\frac{\partial\Phi_2}{\partial z}\quad (\varepsilon_2=2(1-\nu))\end{aligned} \tag{1.3}$$

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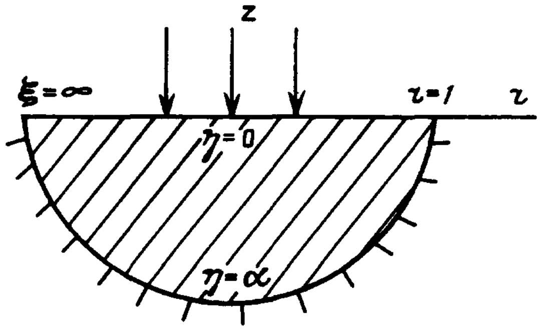

Fig. 1.

and we represent the Boussinesq functions themselves as Mehler–Fock integrals in the complex form [1] ($c > 0$)

$$\Phi_n(\xi, \eta) = \frac{1}{\pi i}\sqrt{\operatorname{ch}\xi + \cos\eta} \int_{c-i\infty}^{c+i\infty} Q_{\mu-1/2}(\operatorname{ch}\xi)[A_n(\mu)\cos\mu\eta + B_n(\mu)\sin\mu\eta]d\mu \ (n = 1, 2) \tag{1.4}$$

$$A_n(-\mu) = -A_n(\mu), \quad B_n(-\mu) = B_n(\mu)$$

By using the formulae

$$\begin{aligned} \frac{\partial}{\partial r} &= \operatorname{sh}\xi\sin\eta\frac{\partial}{\partial\eta} + (1 + \operatorname{ch}\xi\cos\eta)\frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial z} &= (1 + \operatorname{ch}\xi\cos\eta)\frac{\partial}{\partial\eta} - \operatorname{sh}\xi\sin\eta\frac{\partial}{\partial\xi} \end{aligned} \tag{1.5}$$

the differential relation for spherical functions

$$[Q_{\mu-1/2}(\operatorname{ch}\xi)]' = Q^1_{\mu-1/2}(\operatorname{ch}\xi) \tag{1.6}$$

a number of other equations [6] relating spherical functions with different lower an upper indices, the expansion of a δ-function in a Mehler–Fock integral

$$\delta(\xi - x) = \frac{1}{\pi i}\operatorname{sh}x \int_{c-i\infty}^{c+i\infty} \mu P_{\mu-1/2}(\operatorname{ch}x)Q_{\mu-1/2}(\operatorname{ch}\xi)d\mu \tag{1.7}$$

and shifting the integration contour in the integrals (1.4) into the band of regularity of the functions $A_n(\mu), B_n(\mu)$, we change from (1.2)–(1.4) to a system of four functional equations in three straight lines in the plane of the complex variable μ in the unknown functions $A_n(\mu)$, $B_n(\mu)$ ($n = 1, 2$)

$$\begin{aligned} &\varepsilon_2 B_2(\mu) - f(A_1)/2 = W(\mu), \quad \varepsilon_1 A_2(\mu) + f(B_1)/2 = 0 \\ &\cos\mu\alpha[g(A_1)\cos\alpha - g(A_2)\sin\alpha] + \sin\mu\alpha[g(B_1)\cos\alpha - g(B_2)\sin\alpha] + \\ &+(\mu-1)^{-1}\cos(\mu-1)\alpha\{\varepsilon_2 B_2(\mu-1) - (1/2)[A_1(\mu-1) + A_1(\mu)]\} + \\ &+(\mu+1)^{-1}\cos(\mu+1)\alpha\{\varepsilon_2 B_2(\mu+1) + (1/2)[A_1(\mu) + A_1(\mu+1)]\} - \\ &-(\mu-1)^{-1}\sin(\mu-1)\alpha\{\varepsilon_1 A_2(\mu-1) + (1/2)[B_1(\mu-1) + B_1(\mu)]\} - \\ &-(\mu+1)^{-1}\sin(\mu+1)\alpha\{\varepsilon_1 A_2(\mu+1) - (1/2)[B_1(\mu) + B_1(\mu+1)]\} = \\ &= (\mu-1)^{-1}W(\mu-1)\cos(\mu-1)\alpha + (\mu+1)^{-1}W(\mu+1)\cos(\mu+1)\alpha \\ &\varepsilon_2[2A_2(\mu)\cos\alpha\cos\mu\alpha + (\mu-1/2)(\mu-1)^{-1}A_2(\mu-1)\cos(\mu-1)\alpha + \\ &+(\mu+1/2)(\mu+1)^{-1}A_2(\mu+1)\cos(\mu+1)\alpha] + \varepsilon_1[2B_2(\mu)\cos\alpha\sin\mu\alpha + \end{aligned} \tag{1.8}$$

$$+(\mu-\tfrac{1}{2})(\mu-1)^{-1}B_2(\mu-1)\sin(\mu-1)\alpha+(\mu+\tfrac{1}{2})(\mu+1)^{-1}B_2(\mu+1)\sin(\mu+1)\alpha]+$$
$$+f(A_2)\sin\alpha\sin\mu\alpha-f(B_2)\sin\alpha\cos\mu\alpha=-2W(\mu)\cos\alpha\sin\mu\alpha-$$
$$-(\mu-\tfrac{1}{2})(\mu-1)^{-1}W(\mu-1)\sin(\mu-1)\alpha-(\mu+\tfrac{1}{2})(\mu+1)^{-1}W(\mu+1)\sin(\mu+1)\alpha$$
$$f(F)=(\mu-1)F(\mu-1)+2\mu F(\mu)+(\mu+\tfrac{1}{2})F(\mu+1)$$
$$g(F)=F(\mu-1)+2F(\mu)+F(\mu+1)$$
$$W(\mu)=-\operatorname{sh}xP_{\mu-\frac{1}{2}}(\operatorname{ch}x)/(\operatorname{ch}x+1)^{\frac{3}{2}}$$

In the deriving (1.8), the boundary condition for τ_{rz} has been integrated with respect to r (for $\eta = 0$) and, in the boundary condition for σ_z, the first derivative with respect to z has been eliminated using the last relation of (1.6). The third and fourth equations of (1.8) have been written taking account of the first two equations.

We now introduce the new analytic functions $A^*(\mu)$, $B^*(\mu)$ using the formulae

$$A^*(\mu)=\mu[A_1(\mu-\tfrac{1}{2})+A_1(\mu+\tfrac{1}{2})] \qquad (1.9)$$
$$B^*(\mu)=\mu[B_1(\mu-\tfrac{1}{2})+B_1(\mu+\tfrac{1}{2})]$$

The first two equations of (1.8) can then be represented in the form

$$A^*(\mu-\tfrac{1}{2})+A^*(\mu+\tfrac{1}{2})=2[\varepsilon_2B_2(\mu)-W(\mu)]$$
$$B^*(\mu-\tfrac{1}{2})+B^*(\mu+\tfrac{1}{2})=-2\varepsilon_1A_2(\mu) \qquad (1.10)$$

By virtue of the evenness of the function $A^*(\mu)$ and the oddness of the function $B^*(\mu)$ (see (1.4)), the functional equations (1.10) in these functions are solved by reducing them (by putting $\mu = i\tau$, $\tau \in R$) in (1.10)) to Dirichlet problems for the functions $\operatorname{Re} A^*(\mu)$, $\operatorname{Im} B^*(\mu)$ in the band $|\operatorname{Re}\mu| \leqslant 1/2$. The solution of these problems is found using a Fourier integral transformation. The functions $\operatorname{Im} A^*(\mu)$, $\operatorname{Re} B^*(\mu)$ are then recovered using the Schwarz formulae [7]. After Eqs (1.10) have been solved, the functions $A_1(\mu)$, $B_1(\mu)$, which can therefore be eliminated from system (1.8), are determined in a similar manner from relations (1.9).

We now consider system (1.8), having eliminated the functions $A_1(\mu)$, $B_1(\mu)$ from it when $\mu = i\tau$, $\tau \in R$. Using the Schwarz formula, we now reduce the Hilbert boundary-value problem which arises here and has been extended Vekua to a system of two singular integral equations in the new functions $\varphi_1(\tau)$, $\varphi_2(\tau)$ which are related to the functions $A_2(\mu)$, $B_2(\mu)$ by the equations

$$\varphi_1(\tau)=\operatorname{Im}A_2(1+i\tau)+\operatorname{th}\pi\tau\,\omega(\tau)$$
$$\varphi_2(\tau)=\operatorname{Re}B_2(1+i\tau)-\omega(\tau),\quad \omega(\tau)=\operatorname{Re}W(1+i\tau) \qquad (1.11)$$

After some reduction, this system can be written in the matrix form $(0 \leqslant \tau < \infty)$

$$A(\tau)\varphi(\tau)+\frac{1}{\pi i}\int_0^\infty\frac{K(\tau,t)\varphi(\tau)}{\operatorname{ch}\pi t-\operatorname{ch}\pi\tau}d(\operatorname{ch}\pi t)=f(\tau) \qquad (1.12)$$

where $A(\tau) = (a_{nm}(\tau))$, $K(\tau, t) = i(k_{nm}(\tau, t))$ are matrices and $\varphi(\tau) = (\varphi_n(\tau))$, $f(\tau) = (f_n(\tau))$ $(n, m = 1, 2)$ are column vectors. We omit the expressions for the elements of these matrices in view of their length.

System (1.12) has an exact solution when $\alpha = \pi$, $\nu = 1/2$ when $\varphi_1(\tau) = \varphi_2(\tau) \equiv 0$. When $\alpha = \pi$, $\nu \neq 1/2$, the structure of the exact solution of the problem [8, Chapter 12, p. 76] is somewhat more complex. An analysis of this structure and the behaviour of the functions $f_1(\tau)$, $f_2(\tau)$ on the right-hand side of system (1.12) enables one, when $\tau \to \infty$, to assert that the substitution (1.11) separates out the principal parts of the functions $\operatorname{Im} A_2(1 + i\tau)$ and $\operatorname{Re} B_2(1 + i\tau)$ when $\tau \to \infty$.

An investigation of the normality of the system which is characteristic of (1.12), that is, the check that condition [9]

$$\det S_\pm(\tau)=\det[A(\tau)\pm B(\tau)]\neq 0 \quad (0\leqslant\tau<\infty) \qquad (1.13)$$

is satisfied, where $B(\tau) = i(b_{nm}(\tau))$ $(n, m = 1, 2)$, and $b_{nm}(\tau) = k_{nm}(\tau, \tau)$ and $S_\pm(\tau)$ are the basic matrices, shows that the inequality $\nu \neq \cos^2\alpha$ must be satisfied when $[\pi/4, 3\pi/4]$. When $\alpha = \pi n/8$ and $n = 0, 1, 7, 8$, condition (1.14) is satisfied for any value of ν; when $n = 4$, it is satisfied in the interval $\nu > 0.092$; when $n = 2$ and $n = 6$, then $\nu \neq 1/2$ and, when $n = 3, 5$ and $\nu = 0.3$, condition (1.13) is also satisfied.

Using the methods described in [9, 10], we regularize the system of singular equations (1.12), reducing this system to a Fredholm matrix integral equation of the second kind

$$[A_*(\tau)A(\tau) - B_*(\tau)B(\tau)]\varphi(\tau) + \frac{1}{\pi}\int_0^\infty \frac{A_*(\tau)[K(\tau,t) - B(\tau)] + B_*(\tau)[A(t) - A(\tau)]}{\operatorname{ch}\pi t - \operatorname{ch}\pi\tau}\varphi(\tau)d(\operatorname{ch}\pi t) -$$

$$-\frac{1}{\pi^2}B_*(\tau)\int_0^\infty\int_0^\infty \frac{K(t_0,t)}{(\operatorname{ch}\pi t_0 - \operatorname{ch}\pi t)(\operatorname{ch}\pi t_0 - \operatorname{ch}\pi\tau)}\varphi(t)d(\operatorname{ch}\pi t_0)d(\operatorname{ch}\pi t) =$$

$$= A_*(\tau)f(\tau) + \frac{1}{\pi}B_*(\tau)\int_0^\infty \frac{f(t)}{\operatorname{ch}\pi t - \operatorname{ch}\pi\tau}d(\operatorname{ch}\pi t) \tag{1.14}$$

$$A_*(\tau) = (1/2)[S_+^{-1}(\tau) + S_-^{-1}(\tau)], \quad B_*(\tau) = (1/2)[S_+^{-1}(\tau) - S_-^{-1}(\tau)]$$

The investigation of problem (1.1) is completed by solving Eq. (1.14) which can be carried out numerically.

We also note that the numerical solution of the system of singular equations (1.12) can also be obtained without its regularization, by using well-known quadrature formulae for singular integrals [11, 12].

2. We will now study the axially symmetric contact problem of the impression by a force P into the cut of a truncated sphere with an embedded spherical surface of a rigid punch which is circular in plan view. The shape of the base of this punch is described by the function $g_*(\xi)$. Suppose that b is the radius of the circular domain of contact with respect to the coordinate ξ. It is necessary to determine the distribution function of the normal contact pressures under the punch $\sigma_z(\xi, 0) = -\psi(\xi)$ $(0 \leq \xi \leq b)$ and to find the value of P for a specified function $g_*(\xi)$, a specified value of b and the punch δ settling divided by R_*.

After the boundary-value problem (1.1) has been solved, the normal displacement in the cut of the sphere is found using the formula

$$u_z(\xi, 0) = \varepsilon_2\sqrt{\operatorname{ch}\xi + 1}\int_0^\infty \operatorname{th}\pi\tau \operatorname{Im} A_2(i\tau)P_{i\tau-1/2}(\operatorname{ch}\xi)d\tau \tag{2.1}$$

Knowing the function $u_z(\xi, 0)$ of the form of (2.1), taking account of the Schwarz formula

$$\operatorname{Im} A_2(i\tau) = 2\operatorname{sh}\frac{\pi\tau}{2}\int_0^\infty \operatorname{Im} A_2(1 + it)L(t,\tau)dt \tag{2.2}$$

$$L(t,\tau) = \operatorname{sh}(\pi t/2)(\operatorname{ch}\pi t + \operatorname{ch}\pi\tau)^{-1}$$

and the first formula of (1.11), which is a consequence of the Schwarz formula (2.2) for the integral

$$2\operatorname{sh}\frac{\pi\tau}{2}\int_0^\infty L(t,\tau)\operatorname{th}\pi t \operatorname{Re} P_{it+1/2}(\operatorname{ch}x)dt = \operatorname{th}\pi\tau P_{i\tau-1/2}(\operatorname{ch}x) \tag{2.3}$$

the integral equation of the contact problem under consideration in the functions $\psi_*(\xi) = \psi(\xi)/[\theta(\operatorname{ch}\xi + 1)^{3/2}]$ $(\theta = G/(1-\nu))$ can be written in the form

$$\int_0^b \psi_*(x)\operatorname{sh}x\,dx\{\int_0^\infty [(1 - g(\tau))P_{i\tau-1/2}(\operatorname{ch}x) +$$

$$+ 2\operatorname{th}\pi\tau\operatorname{sh}\frac{\pi\tau}{2}\int_0^\infty \varphi_1^*(t,x)L(t,\tau)dt]P_{i\tau-1/2}(\operatorname{ch}\xi)d\tau\} = f_*(\xi) \tag{2.4}$$

Here, $0 \leq \xi \leq b$ and we have introduced the notation

$$g(\tau) = 1/\operatorname{ch}^2\pi\tau, \quad \varphi_1^*(t,x)\operatorname{sh}x = \varphi_1(t)(\operatorname{ch}x + 1)^{3/2} \tag{2.5}$$

$$f_*(\xi) = [\delta - g_*(\xi)] / \sqrt{\operatorname{ch}\xi + 1}$$

From mechanical considerations, the kernel of the integral equation (2.4) is symmetric. Interchanging the variables x and ξ in this kernel we write the paired integral equation, which is equivalent to Eq. (2.4), in the form

$$\int_0^\infty C(\tau)\{[1 - g(\tau)]P_{i\tau-\frac{1}{2}}(\operatorname{ch}\xi) + 2\operatorname{th}\pi\tau\operatorname{sh}\frac{\pi\tau}{2}\int_0^\infty \varphi_1^*(t,\xi)L(t,\tau)dt\}d\tau = f_*(\xi) \quad (0 \leqslant \xi \leqslant b) \tag{2.6}$$

$$\int_0^\infty C(\tau)\tau\operatorname{th}\pi\tau P_{i\tau-\frac{1}{2}}(\operatorname{ch}\xi)d\tau = 0 \quad (b < \xi < \infty)$$

where

$$C(\tau) = \int_0^b \psi_*(x)\operatorname{sh}x P_{i\tau-\frac{1}{2}}(\operatorname{ch}x)dx \tag{2.7}$$

On finding the function $C(\tau)$ in the form of the integral

$$C(\tau) = \int_0^b \varphi_*(x)\cos\tau x\, dx \tag{2.8}$$

and using a well-known technique [13, p. 101] and the value of the integral (the Schwarz formula again)

$$2\operatorname{sh}\frac{\pi\tau}{2}\int_0^\infty L(t,\tau)\operatorname{th}\pi t\cos tx\, dt = \frac{\operatorname{th}\pi\tau\cos\tau x}{\operatorname{ch}x} \tag{2.9}$$

we reduce the paired equation (2.6) to a Fredholm integral equation of the second kind in the function $\varphi_*(x)$

$$\varphi_*(x) - \frac{1}{\pi}\int_0^b \varphi_*(y)K_*(x,y)dy = F(x) \quad (0 \leqslant x \leqslant b) \tag{2.10}$$

$$K_*(x,y) = \frac{1}{2\pi}\left[\frac{x+y}{\operatorname{sh}((x+y)/2)} + \frac{x-y}{\operatorname{sh}((x-y)/2)}\right] - K_1(x,y)$$

$$K_1(x,y) = \frac{2}{\operatorname{ch}y}\frac{d}{dx}\int_0^x\int_0^\infty \frac{\varphi_1^*(t,\xi)\operatorname{sh}\xi\operatorname{th}\pi t}{\sqrt{2(\operatorname{ch}x - \operatorname{ch}\xi)}}\cos ty\, dt\, d\xi \tag{2.11}$$

$$F(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{f_*(\xi)\operatorname{sh}\xi}{\sqrt{2(\operatorname{ch}x - \operatorname{ch}\xi)}}d\xi$$

Using the formula

$$\frac{d}{dx}\int_0^x \frac{\operatorname{Re}P_{i\tau+\frac{1}{2}}(\operatorname{ch}\xi)\operatorname{sh}\xi}{\sqrt{2(\operatorname{ch}x - \operatorname{ch}\xi)}}d\xi = \operatorname{ch}x\cos\tau x \tag{2.12}$$

which follows from representation 8.715 in [6] and Abel's inversion formula, expression (2.11) for the function $K_1(x, y)$ can be simplified to the integral

$$K_1(x,y) = 2\frac{\operatorname{ch}x}{\operatorname{ch}y}\int_0^\infty \Phi_1^*(t,x)\operatorname{th}\pi t\cos ty\, dt \tag{2.13}$$

where the function $\Phi_1^*(\tau, x) = \varphi_1(\tau)$ satisfies system (1.14) where, instead of the function $\omega(\tau)$ of the form of (1.11), we have put $\omega_*(\tau) = \cos\tau x$ on the right-hand side of this system.

The mechanical meaning of the function $\varphi_*(x)$ lies in the fact that its value at the point $x = b$ characterizes the coefficient for the singularity of the function of the required contact pressures $\psi_*(x)$ at this very point. In fact, using formula (2.7), we obtain the function $\psi_*(x)$ in the form

Table 1

b	χ						$\varkappa$
	$n = 1$	2	3	4	5	6	
0.1	300	302	312	324	324	325	0.050
1	240	146	194	347	350	359	0.462
2	94.3	75.7	127	310	344	359	0.762

$$\psi_*(x) = \int_0^\infty C(\tau)\tau \operatorname{th} \pi\tau \, P_{i\tau-\frac{1}{2}}(\operatorname{ch} x)d\tau \tag{2.14}$$

Substituting its representation (2.8), instead of the function $C(\tau)$, into (2.14) and integrating by parts, we find that

$$\lim_{x\to b} \sqrt{\operatorname{ch} b - \operatorname{ch} x}\, \psi_*(x) = \varphi_*(b) / \sqrt{2} \tag{2.15}$$

The values of the quantity $\chi = 10^3 \times \varphi_*(b)/(\sqrt{(2)}\delta)$ and the ratio $\varkappa$ of the dimensional radius of the circular domain of contact to the radius of cross section R_* for the case when $\nu = 0.3, g_*(\xi) \equiv 0$ (a plane punch, $F(x) = \sqrt{(2)}\delta/[\pi \operatorname{ch}(x/2)]$) are given in Table 1 for different b and angles $\alpha = \pi n/6$, which characterize the degree of truncation of the sphere.

Calculations show that, as b increases, the value of $\varphi_*(b)$ can change sign, that is, the elastic medium of the sphere can depart from the edge of the punch. This occurs, for example, when $b = 3$ (the dimensional radius of the domain of contact is equal to $0.905R_*$) and $n = 1, 3$ and 6.

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